

Integrable Structure of the Dirichlet Boundary Problem in Two Dimensions

A.Marshakov ^{*} P.Wiegmann [†] A.Zabrodin [‡]

September 2001

Abstract

We study how the solution of the two-dimensional Dirichlet boundary problem for smooth simply connected domains depends upon variations of the data of the problem. We show that the Hadamard formula for the variation of the Dirichlet Green function under deformations of the domain reveals an integrable structure. The independent variables corresponding to the infinite set of commuting flows are identified with harmonic moments of the domain. The solution to the Dirichlet boundary problem is expressed through the tau-function of the dispersionless Toda hierarchy. We also discuss a degenerate case of the Dirichlet problem on the plane with a gap. In this case the tau-function is identical to the partition function of the planar large N limit of the Hermitean one-matrix model.

^{*}Theory Department, Lebedev Physics Institute, Leninsky pr. 53, 117924 Moscow, Russia and ITEP, Bol. Cheremushkinskaya str. 25, 117259 Moscow, Russia

[†]James Franck Institute and Enrico Fermi Institute of the University of Chicago, 5640 S.Ellis Avenue, Chicago, IL 60637, USA and Landau Institute for Theoretical Physics, Moscow, Russia

[‡]Institute of Biochemical Physics, Kosygina str. 4, 119991 Moscow, Russia and ITEP, Bol. Cheremushkinskaya str. 25, 117259 Moscow, Russia

1 Introduction

The subject of the Dirichlet boundary problem in two dimensions [1] is a harmonic function in a domain of the complex plane bounded by a closed curve with a given value on the boundary and continuous up to the boundary. The question we address in this paper is how the harmonic function in the bulk varies under a small deformation of the the shape of the domain.

Remarkably, this standard problem of complex analysis possesses an integrable structure [2, 3] which we intend to clarify further in this paper. It is described by a particular solution of an integrable hierarchy of partial differential equations known in the literature as dispersionless Toda (dToda) hierarchy. Moreover, related integrable hierarchies arise in the context of 2D topological theories and just the same solution to the dToda hierarchy emerges in the study of 2D quantum gravity [4, 5] (we do not elaborate these relations in this paper).

Let D be a simply connected domain in the complex plane bounded by a smooth simple curve γ . The Dirichlet problem is to find a harmonic function $u(z)$ in D such that it is continuous up to the boundary and equals a given function $u_0(z)$ on the boundary. The problem has a unique solution written in terms of the Green function $G(z_1, z_2)$ of the Dirichlet boundary problem:

$$u(z) = -\frac{1}{2\pi} \oint_{\gamma} u_0(\xi) \partial_n G(z, \xi) |d\xi|, \quad (1.1)$$

where ∂_n is the normal derivative on the boundary with respect to the second variable, and the normal vector \vec{n} always looks *inside* the domain, where the Dirichlet problem is posed. Equivalently, the solution is represented as $u(z) = \frac{1}{\pi i} \oint_{\partial D} u_0(\xi) \partial_{\xi} G(z, \xi) d\xi$, where ∂D is understood as γ run anticlockwise with respect to the domain.

The main object to study is, therefore, the Dirichlet Green function. It is uniquely determined by the following properties [1]:

- (G1) The function $G(z_1, z_2) - \log |z_1 - z_2|$ is symmetric, bounded and harmonic everywhere in D in both arguments;
- (G2) $G(z_1, z_2) = 0$ if any one of the variables belongs to the boundary.

The definition implies that $G(z_1, z_2)$ is real and negative in D . The Green function can be written explicitly through a conformal map of the domain D onto some “reference” domain for which the Green function is known. A convenient choice is the unit disk. Let $f(z)$ be any bijective conformal map of D onto the unit disk (or its complement), then

$$G(z_1, z_2) = \log \left| \frac{f(z_1) - f(z_2)}{f(z_1) \overline{f(z_2)} - 1} \right|, \quad (1.2)$$

where bar means complex conjugation. Such a map exists by virtue of the Riemann mapping theorem [1].

It thus suffices to study variations of the conformal map $f(z)$ under deformations of the boundary. This problem was discussed in [2, 3], where it was shown that evolution

of the conformal map under changing harmonic moments of the domain is given by the dToda integrable hierarchy¹. The study of the Dirichlet problem approaches this subject from another angle.

Our starting point is the Hadamard variational formula [7]. It gives variation of the Green function under small deformations of the domain in terms of the Green function itself:

$$\delta G(z_1, z_2) = \frac{1}{2\pi} \oint_{\gamma} \partial_n G(z_1, \xi) \partial_n G(z_2, \xi) \delta h(\xi) |d\xi|. \quad (1.3)$$

Here $\delta h(\xi)$ is the thickness between the curve γ and the deformed curve, counted along the normal vector at the point $\xi \in \gamma$. We show that already this remarkable formula reflects all integrable properties of the Dirichlet problem.

A smooth closed curve γ (for simplicity, we may assume it to be analytic in order to have an easy sufficient justification of some arguments below) divides the complex plane into two parts having the common boundary: a compact interior domain D_{int} , and an exterior domain D_{ext} containing ∞ . Correspondingly, one recognizes *interior* and *exterior* Dirichlet problems. The main contents of the paper is common for both of them. To stress this, we try to keep the notation uniform calling the domain simply D . We will show that (logarithms of) the tau-functions, introduced in [2, 3] and further studied in [8], for the interior and exterior problems are related to each other by a Legendre transform.

The exterior Dirichlet problem makes sense when the interior domain degenerates into a segment (a plane with a gap). We will show that in this case a deformation problem is described by the dispersionless limit of the Toda chain hierarchy and discuss its relation to the planar limit of the Hermitean matrix model.

2 Deformations of the boundary

Let D be a simply-connected domain in the extended complex plane bounded by a smooth simple curve γ . Consider a basis $\psi_k(z)$, $k \geq 1$, of holomorphic functions in D such that $\psi_k(z_0) = 0$ for some point $z_0 \in D$. We call z_0 *the normalization point*. The basis is assumed to be fixed and independent of the domain. For example, in case of the interior problem one may assume, without loss of generality, that the origin is in D and set $z_0 = 0$, $\psi_k(z) = z^k/k$, while a natural choice for the exterior problem is $z_0 = \infty$ and $\psi_k(z) = z^{-k}/k$. Throughout the paper, these bases for interior and exterior problems are referred to as natural ones.

Let t_k be moments of the domain D defined with respect to the basis ψ_k :

$$t_k = \kappa \frac{1}{\pi} \int_D \psi_k(z) d^2 z, \quad k = 1, 2, \dots \quad (2.1)$$

where $\kappa = \pm$ for the interior (exterior) problem. We also assume that the functions ψ_k for domains containing ∞ are integrable, or the integrals are properly regularized (see below). Besides, we denote by t_0 the area (divided by π) of the domain D in case of the

¹A relation between conformal maps (of slit domains) and special solutions to some integrable equations of hydrodynamic type was earlier observed by Gibbons and Tsarev [6].

interior problem and that of the complementary (compact) domain in case of the exterior problem:

$$t_0 = \begin{cases} \frac{1}{\pi} \int_D d^2 z & \text{for compact domains} \\ \frac{1}{\pi} \int_{\mathbb{C} \setminus D} d^2 z & \text{for non-compact domains} \end{cases}$$

Let us note that the moments (except for t_0) are in general complex. We call the quantities t_k , \bar{t}_k and t_0 *harmonic moments* of the domain D . The Stokes formula represents the harmonic moments as contour integrals

$$t_k = \frac{1}{2\pi i} \oint_{\gamma} \psi_k(z) \bar{z} dz, \quad k = 0, 1, 2, \dots$$

(where it is set $\psi_0(z) = 1$) providing, in particular, a regularization of possibly divergent integrals (2.1) in case of the exterior problem. Throughout the paper the contour in \oint_{γ} is run in anticlockwise direction both for interior and exterior problems.

The basic fact of the theory of deformations of closed analytic curves is that the (in general complex) moments t_k supplemented by the real variable t_0 form a set of local coordinates in the space of smooth closed curves [9] (see also [10]). This means that under any small deformation of the domain the set $\{t_0, t_1, \dots\}$ is subject to a small change and vice versa. More precisely, let $\gamma(t)$ be a family of curves such that $\partial_t t_k = 0$ in some neighborhood of $t = 0$, then all the curves $\gamma(t)$ coincide with $\gamma = \gamma(0)$ in this neighborhood.

The family of differential operators

$$\nabla(z) = \partial_{t_0} + \sum_{k \geq 1} \left(\psi_k(z) \partial_{t_k} + \overline{\psi_k(z)} \partial_{\bar{t}_k} \right) \quad (2.2)$$

span the complexified tangent space to the space of curves. They are invariant under change of variables in the following sense: let \tilde{t}_k be harmonic moments defined with respect to another basis, $\tilde{\psi}_k$, of holomorphic functions in D ; then $\tilde{\nabla}(z) = \nabla(z)$. Note that $\nabla(z_0) = \partial_{t_0}$ since $\psi_k(z_0) = 0$. The operator $\nabla(z)$ has a clear geometrical meaning described below.

Let us consider a special deformation of the domain obtained by adding to it an infinitesimal smooth bump (of an arbitrary form) with area ϵ located at the point $\xi \in \gamma$. Our convention is that $\epsilon > 0$ if the bump looks outside the domain in which the Dirichlet problem is posed, as is shown in Fig. 1.

Let A be any functional of a domain that depends on the harmonic moments only. The variation of such a functional in the leading order in ϵ , is given by

$$\delta_{\epsilon(\xi)} A = \kappa \frac{\epsilon}{\pi} \nabla(\xi) A, \quad \xi \in \gamma, \quad (2.3)$$

Indeed, combining $\delta A = \partial_{t_0} A \delta t_0 + \sum_{k \geq 1} (\partial_{t_k} A \delta t_k + \partial_{\bar{t}_k} A \delta \bar{t}_k)$ and

$$\delta t_k = \kappa \frac{1}{\pi} \int_{\text{bump}} \psi_k(z) d^2 z = \kappa \frac{\epsilon}{\pi} \psi_k(\xi)$$

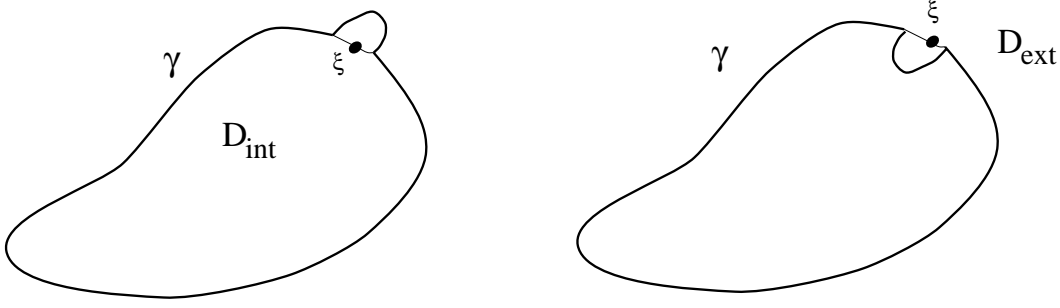


Figure 1: Action of the operator $\nabla(\xi)$ in the case of interior D_{int} (left) and exterior D_{ext} (right) domains. In our convention bump always looks outside the (interior or exterior) domain.

we obtain (2.3). So, the result of the action of the operator $\nabla(\xi)$ with $\xi \in \gamma$ on A is proportional to the variation of the functional under attaching a bump at the point ξ . To put it differently, we can say that the boundary value of the function $\nabla(z)A$ is given by the l.h.s. of (2.3). For functionals A such that the series $\nabla(z)A$ converges everywhere in D up to the boundary, this remark gives a usable method to find the function $\nabla(z)A$ everywhere in the domain. This function is harmonic in D with the boundary value determined from (2.3). It is given by (1.1):

$$\frac{\epsilon}{\pi} \nabla(z)A = \kappa \frac{1}{2\pi} \oint_{\gamma} |d\xi| \partial_n G(z, \xi) \delta_{\epsilon(\xi)} A. \quad (2.4)$$

This gives the result of the action of the operator $\nabla(z)$, when the argument is anywhere in D .

For example, given any regular function f in a domain containing the interior domain D , set $A_f = \int_D f(z) d^2 z$. We have: $\delta_{\epsilon(\xi)} A_f = \frac{\epsilon}{\pi} \nabla(\xi) A_f = \epsilon f(\xi)$, $\xi \in \gamma$. If the function f is harmonic in D , then $\nabla(z) A_f = \pi f(z)$ for any $z \in D$.

The subject of the deformation theory of the Dirichlet boundary problem is to compute $\nabla(z)G(z_1, z_2)$ through the conformal map or the Green function of the original domain. In the next section, we do this using the Hadamard variational formula.

3 Hadamard variational formula and dispersionless integrable hierarchy

3.1 The Hadamard integrability condition

Variation of the Green function under small deformations of the domain is known due to Hadamard [7], see eq. (1.3). Being specified to the particular case of attaching a bump of the area ϵ , it reads:

$$\delta_{\epsilon(\xi)} G(z_1, z_2) = -\frac{\epsilon}{2\pi} \partial_n G(z_1, \xi) \partial_n G(z_2, \xi), \quad \xi \in \gamma. \quad (3.1)$$

To find how the Green function changes under a variation of the harmonic moments, we use (2.4) to employ the harmonic continuation procedure explained in the previous

section. The harmonic function in D with a boundary value given by the Hadamard formula is

$$\nabla(z_3)G(z_1, z_2) = \kappa \frac{1}{4\pi} \oint_{\gamma} \partial_n G(z_1, \xi) \partial_n G(z_2, \xi) \partial_n G(z_3, \xi) |d\xi| \quad (3.2)$$

It is obvious from the r.h.s. of (3.2) that the result of the action of the operator $\nabla(z)$ on the Green function is harmonic and symmetric in all three arguments, i.e.,

$$\nabla(z_3)G(z_1, z_2) = \nabla(z_1)G(z_2, z_3). \quad (3.3)$$

This is our basic relation. It has the form of integrability condition. In the rest of the paper we will draw consequences of this symmetry and underlying algebraic structures. Note also that despite the Green function vanishes on the boundary, its derivative (the l.h.s. of eq. (3.3)) with respect to the deformation of the domain does not.

The basic equation (3.3) is a compressed form of an integrable hierarchy. To unfold it, let us separate holomorphic and antiholomorphic parts of this equation.

Let E be the exterior to the unit disk. Given a point $a \in D$, consider a bijective conformal map $f_a : D \rightarrow E$ such that $f_a(a) = \infty$. The Dirichlet Green function then is

$$G(a, z) = -\log |f_a(z)|. \quad (3.4)$$

Under a proper normalization of the map the integrability condition (3.3) becomes holomorphic:

$$\nabla(b) \log f_a(z) = \nabla(a) \log f_b(z) \quad (3.5)$$

for all $a, b, z \in D$.

The following normalization will be convenient: the overall phase is chosen to be $\arg f_a(z_0) = \pi - \arg(z_0 - a)$ if $a \neq z_0$, where $z_0 \in D$ is the normalization point. If $a = z_0$ we set $\lim_{z \rightarrow z_0} ((z - z_0)^2 f'(z))$ to be real and negative. Under these conditions

$$f_a(z) = \left(\frac{(\bar{a} - \bar{z}_0) f(a)}{(a - z_0) \overline{f(a)}} \right)^{1/2} \frac{f(z) \overline{f(a)} - 1}{f(a) - f(z)} \quad (3.6)$$

(for $a, z_0 \neq \infty$). In the vicinity of the point a ($a \neq \infty$)

$$f_a(z) = e^{i\omega(a, z_0)} \frac{r_a}{z - a} \left(1 + \sum_{k \geq 1} p_k(a) (z - a)^k \right), \quad (3.7)$$

where the real constant r_a is called conformal radius of the domain [11] with respect to the point a , and ω is a phase determined from the normalization condition. In particular, they read $\omega(z_0, z_0) = 0$. Similarly, the map $f_a(z)$ can be defined in the case when either a or z_0 lies at infinity.

To verify (3.5), we note that the holomorphic function $\nabla(b) \log f_a(z) - \nabla(a) \log f_b(z)$ is also antiholomorphic (in z) by virtue of (3.3), and thus must be a constant. Setting $z = z_0$ we find that the latter is zero:

$$\begin{aligned} & \nabla(b) \log |f_a(z_0)| - \nabla(a) \log |f_b(z_0)| + \\ & + i \nabla(b) \arg f_a(z_0) - i \nabla(a) \arg f_b(z_0) = 0 \end{aligned}$$

(the first line vanishes due to (3.3), the second one vanishes because the normalization does not depend on the shape of the domain).

3.2 Harmonic moments as commuting flows

Equation (3.5) suggests to treat $\log f_a(z)$ as a generating function of commuting flows with respect to spectral parameter a . The expansion

$$\log f_a(z) = H_0(z) + \sum_{k \geq 1} \left(\psi_k(a) H_k(z) - \overline{\psi_k(a)} \tilde{H}_k(z) \right) \quad (3.8)$$

defines generators H_k, \tilde{H}_k of the commuting flows. Clearly, $H_0(z) = \log f(z)$. It implies evolution equations for $f(z)$,

$$\frac{\partial \log f(z)}{\partial t_k} = \frac{\partial H_k(z)}{\partial t_0}, \quad \frac{\partial \log f(z)}{\partial \bar{t}_k} = -\frac{\partial \tilde{H}_k(z)}{\partial t_0}$$

and integrability conditions:

$$\frac{\partial H_k(z)}{\partial t_j} = \frac{\partial H_j(z)}{\partial t_k}, \quad \frac{\partial \tilde{H}_k(z)}{\partial \bar{t}_j} = \frac{\partial \tilde{H}_j(z)}{\partial \bar{t}_k}, \quad \frac{\partial H_k(z)}{\partial \bar{t}_j} = -\frac{\partial \tilde{H}_j(z)}{\partial t_k}. \quad (3.9)$$

The real part of (3.8) vanishes on the boundary (as it is the Dirichlet Green function), therefore, the boundary values of \tilde{H}_k and H_k are complex conjugated:

$$\tilde{H}_k(z) = \overline{H_k(z)}, \quad z \in \gamma. \quad (3.10)$$

The structure of integrable hierarchy becomes explicit if instead of functions of z one passes to functions of its image w under the map f_{z_0} : $w = f_{z_0}(z) \equiv f(z)$.

Using the chain rule, one can write

$$\nabla(a) \log f_b(z) = \nabla(a) \log f_b(z(w)) \Big|_w + (\nabla(a) \log f(z)) w \partial_w \log f_b.$$

In the last term we observe that $\nabla(a) \log f(z) = \partial_{t_0} \log f_a(z)$ (using (3.5) at $b = z_0$). Subtracting the same equality with a, b interchanged, we come to equation of the zero-curvature type:

$$\nabla(a) \log f_b - \nabla(b) \log f_a - \{\log f_a, \log f_b\} = 0,$$

where the Poisson brackets are defined as $\{f, g\} \equiv w \frac{\partial f}{\partial w} \frac{\partial g}{\partial t_0} - w \frac{\partial g}{\partial w} \frac{\partial f}{\partial t_0}$ and t_k -derivatives are taken now at fixed w .

Let $z(w)$ be the map inverse to $w = f(z)$. Equation (3.5) at $b = z_0$, being rewritten for the inverse map, has the form of a one-parametric family of evolution equations of the Lax type labeled by the spectral parameter a . They are

$$\nabla(a) z(w) = \{\log f_a(z(w)), z(w)\}. \quad (3.11)$$

We refer to them as to *deformation equations*. The zero-curvature conditions ensure that these equations are consistent, i.e. flows with different values of spectral parameter commute.

The integrability conditions (3.9) in the new variable acquire the form of the zero-curvature equations

$$\begin{aligned}\partial_{t_j} H_i(w) - \partial_{t_i} H_j(w) + \{H_i(w), H_j(w)\} &= 0, \\ \partial_{\tilde{t}_j} \tilde{H}_i(w) - \partial_{\tilde{t}_i} \tilde{H}_j(w) + \{\tilde{H}_j(w), \tilde{H}_i(w)\} &= 0, \\ \partial_{t_j} \tilde{H}_i(w) + \partial_{\tilde{t}_i} H_j(w) + \{\tilde{H}_i(w), H_j(w)\} &= 0.\end{aligned}\tag{3.12}$$

From (3.6) it is easy to see that H_k are polynomials in w while \tilde{H}_k are polynomials in w^{-1} . Furthermore, for any basis such that $\psi_k(z) = O((z-z_0)^k)$ these polynomials are of degree k . In other words, the generators are meromorphic functions on the Riemann sphere with two marked points at $w = 0$ and $w = \infty$. This is a particular case of the universal Whitham hierarchy [12], known as dispersionless Toda lattice. In [13], it is proved that the full set of zero-curvature conditions (3.12), together with the polynomial structure of the generators, already imply existence of the Lax function and Lax-Sato equations.

For a particular choice of the basis, the Lax function can be expressed through the inverse conformal map. Consider the interior problem, set $z_0 = 0 \in D$ and fix the basis of holomorphic functions in D to be the natural one: $\psi_k(z) = z^k/k$. From (3.7) and (3.8) it follows that \tilde{H}_k are holomorphic in D while H_k are meromorphic with the k -th order pole at 0 so that $H_k = z^{-k} + O(1)$ as $z \rightarrow 0$. Combining these properties with the polynomial structure of the generators as functions of w , and taking into account (3.10), one gets

$$\begin{aligned}H_k &= \left(z^{-k}(w)\right)_{>0} + \frac{1}{2} \left(z^{-k}(w)\right)_0, \\ \tilde{H}_k &= \left(\bar{z}^{-k}(w^{-1})\right)_{<0} + \frac{1}{2} \left(\bar{z}^{-k}(w^{-1})\right)_0,\end{aligned}\tag{3.13}$$

where $z(w)$ is the map inverse to $f(z)$ and $\bar{z}(w) = \overline{z(\bar{w})}$ is its Schwarz double. The symbols $(f(w))_{>0}$, $(f(w))_{<0}$ mean truncated Laurent series, where only terms with strictly positive (negative) powers of w are kept, while $(f(w))_0$ is the free term (w^0) of the series. The free terms in (3.13) are found with the help of an easily proved identity $\log \frac{w(z)}{z} = \sum_{k \geq 1} \frac{1}{k} (z^{-k}(w))_0 z^k$ valid for any series of the form $w(z) = z + w_1 z^2 + \dots$. Similar formulas hold for the exterior problem with the choice $z_0 = \infty$, $\psi_k(z) = z^{-k}/k$.

3.3 Deformation equations and dispersionless Toda hierarchy

Expanding the deformation equations (3.11) in spectral parameter we obtain the Lax representation of the dToda hierarchy. To see this, we recall that the dToda hierarchy is an infinite set of evolution (Lax-Sato) equations for two Lax functions

$$\begin{aligned}L(w) &= rw + \sum_{k \geq 0} u_k w^{-k}, \\ \tilde{L}(w) &= rw^{-1} + \sum_{k \geq 0} \tilde{u}_k w^k.\end{aligned}$$

The equations are

$$\frac{\partial L(w)}{\partial t_k} = \{H_k(w), L(w)\}, \quad \frac{\partial L(w)}{\partial \tilde{t}_k} = \{L(w), \tilde{H}_k(w)\}$$

and the same for $\tilde{L}(w)$. Here the Poisson brackets with respect to w and t_0 are defined as above, and

$$H_k(w) = \left(L^k(w)\right)_{>0} + \frac{1}{2}\left(L^k(w)\right)_0$$

$$\tilde{H}_k(w) = \left(\tilde{L}^k(w)\right)_{<0} + \frac{1}{2}\left(\tilde{L}^k(w)\right)_0$$

are generators of the flows. One may also introduce $H_0(w) = \log w$. They obey the dispersionless zero-curvature conditions (3.12) which express consistency of the Lax equations and generate an infinite set of partial differential equations for coefficients of the Lax functions.

To summarize, we have the following identification of the Lax functions with conformal maps:

- For the interior problem ($z_0 = 0$ with the natural basis):

$$L(w) = \frac{1}{z(w)}, \quad \tilde{L}(w) = \frac{1}{\bar{z}(w^{-1})}, \quad (3.14)$$

where $z(w)$ is the function inverse to $w = f_0(z)$;

- For the exterior problem ($z_0 = \infty$ with the natural basis):

$$L(w) = z(w), \quad \tilde{L}(w) = \bar{z}(w^{-1}), \quad (3.15)$$

where $z(w)$ is the function inverse to $w = f_\infty(z)$

So the inverse map, $z(w)$, and its Schwarz double, $\bar{z}(w^{-1})$, both obey the Lax equations.

Let us comment on another choice of basis. Suppose λ^{-1} is a local parameter at the normalization point z_0 , i.e.,

$$\lambda = \frac{1}{z - z_0} + c_0 + c_1(z - z_0) + O((z - z_0)^2), \quad z_0 \neq \infty$$

$$\lambda = z + c_0 + \frac{c_1}{z} + O(z^{-2}), \quad z_0 = \infty.$$

with some domain-independent coefficients c_j . Assuming that $\lambda^{-1}(z)$ is a well defined local parameter in a domain that contains D , we choose the basis of holomorphic functions in D to be $\psi_k(z) = \lambda^{-k}(z)/k$. This results in a linear change of times with the help of a triangular matrix. Repeating the above arguments, one identifies the Lax function with $\lambda(z)$: $L(w) = \lambda(z(w))$. This Lax function provides a solution to an equivalent hierarchy in the sense of [14].

3.4 String equations

A customary way to fix a solution to a dispersionless hierarchy is to impose an additional constraint on the Lax functions (sometimes called *string equation* [15]). We are going to show that not only the deformation equations but the string equation, too, is an easy consequence of the Hadamard formula and our basic relation (3.3).

By $r = r_{z_0}$ denote the conformal radius of the curve γ with respect to the normalization point z_0 . Let us calculate $\delta_{\epsilon(\xi)} \log r$ in two different ways. The first one is to use the Hadamard formula (3.1) when both arguments tend to the normalization point:

$$\delta_{\epsilon(\xi)} \log r = \frac{2\epsilon}{\pi} |\partial_{\xi} G(z_0, \xi)|^2 = \frac{\epsilon}{2\pi} |\partial_{\xi} f(\xi)|^2$$

(recall that $|f(\xi)| = 1$ for $\xi \in \gamma$). The second one is obtained from (3.3) in the limit $z_1 \rightarrow \xi \in \gamma, z_2, z_3 \rightarrow z_0$:

$$\delta_{\epsilon(\xi)} \log r = -\frac{\epsilon}{\pi} \partial_{t_0} \log |f(\xi)|,$$

where we have used (2.3). Combining the results, we obtain the relation

$$\partial_{t_0} \log |f(z)|^2 = -|\partial_z f(z)|^2, \quad z \in \gamma.$$

Passing to the variable $w = f(z)$, one rewrites this equation as

$$2\mathcal{R}e \left(w \frac{\partial z(w)}{\partial w} \frac{\partial \overline{z(w)}}{\partial t_0} \right) = 1, \quad |w| = 1,$$

where $z(w)$ is the inverse map, as before. Being analytically continued from the unit circle, it reads

$$\{z(w), \bar{z}(w^{-1})\} = 1.$$

This is the customary form of the semiclassical string equation. Similar arguments in case of the exterior problem lead to the same equation.

For the natural basis, let us rewrite the string equation in terms of the Lax functions L, \tilde{L} . Using the identifications (3.14) and (3.15), we have:

$$\begin{aligned} \{L^{-1}(w), \tilde{L}^{-1}(w)\} &= -1 && \text{for the interior problem} \\ \{L(w), \tilde{L}(w)\} &= 1 && \text{for the exterior problem} \end{aligned} \tag{3.16}$$

So, although the string equation in terms of $z(w)$ is the same, the interior and exterior problems correspond to two different solutions of the same (dToda) integrable hierarchy (cf. [16]).

4 Tau-function and dispersionless Hirota equations

4.1 Tau-function

Symmetry relation (3.3) implies that there exists a real-valued function of harmonic moments $F(t_0, \mathbf{t}, \bar{\mathbf{t}}) = F(t_0; t_1, t_2, \dots; \bar{t}_1, \bar{t}_2, \dots)$ such that

$$G(z_1, z_2) = g_0(z_1, z_2) + \frac{1}{2} \nabla(z_1) \nabla(z_2) F,$$

where the function g_0 does not depend on the domain, i.e., on moments (the coefficient $\frac{1}{2}$ is set for future convenience). Note that adding to F a quadratic form in t_k, \bar{t}_k amounts to changing the function g_0 only.

From the definition of the Green function it follows that $g_0(z_1, z_2)$ is symmetric and harmonic in D in both arguments with the only singularity $\log |z_1 - z_2|$ as $z_1 \rightarrow z_2$, or, in terms of a local parameter λ^{-1} , $\log |\lambda^{-1}(z_1) - \lambda^{-1}(z_2)|$. In case of the interior problem, D can be any domain not containing the point of infinity. The function g_0 must be the same for all such domains. Therefore, singularities of g_0 in the extended complex plane, other than the logarithmic singularity at merging points, may occur only at infinity. In other words, g_0 is of the form $g_0(z_1, z_2) = \log |\lambda^{-1}(z_1) - \lambda^{-1}(z_2)| + h_0(z_1, z_2)$ where h_0 is regular and harmonic in both arguments everywhere in the complex plane but at infinity. The function h_0 is in fact a matter of definition of the tau-function. For some particular basis we are free to choose it to be zero. Hence we obtain the following important relation:

$$G(z_1, z_2) = \log |\lambda^{-1}(z_1) - \lambda^{-1}(z_2)| + \frac{1}{2} \nabla(z_1) \nabla(z_2) F, \quad z_1, z_2 \in D \quad (4.1)$$

Changing the normalization point or passing to another local parameter results in modifying the function g_0 . As is already pointed out, this is equivalent to adding to F a quadratic form in times. Taking into account the remark at the end of Sec. 3.4, one sees that this agrees with the relation between tau-functions of equivalent hierarchies discussed in [17]. It is easy to see that in this form the relation holds true for the exterior problem as well.

For clarity, let us specify the above equation for the natural basis. Instead of the differential operator $\nabla(z)$ defined for arbitrary basis, it is convenient to use its specialization for the natural basis

$$\mathcal{D}(z) = \partial_{t_0} + \sum_{k \geq 1} \left(\frac{z^{-k}}{k} \partial_{t_k} + \frac{\bar{z}^{-k}}{k} \partial_{\bar{t}_k} \right) \quad (4.2)$$

so that $\nabla(z) = \mathcal{D}(z)$ for the exterior problem and $\nabla(z) = \mathcal{D}(z^{-1})$ for the interior one. Eq. (4.1) is then rewritten in the form

$$G(z_1, z_2) = \begin{cases} \log |z_1 - z_2| + \frac{1}{2} \mathcal{D}(z_1^{-1}) \mathcal{D}(z_2^{-1}) F & \text{for the interior problem} \\ \log |z_1^{-1} - z_2^{-1}| + \frac{1}{2} \mathcal{D}(z_1) \mathcal{D}(z_2) F & \text{for the exterior problem} \end{cases} \quad (4.3)$$

The function F plays a central role in what follows. It is the tau-function of curves discussed in [2, 3, 8]. It is a dispersionless limit of logarithm of tau-function of an integrable hierarchy or *dispersionless tau-function* [12]. For brevity we refer to it simply as tau-function. A relation between the tau-function and the Green function of the type (4.3) has been emphasized by L. Takhtajan [10]. In sect. 5 we give a few explicit integral representations for the tau-function.

4.2 Conformal maps and tau-function

In this section, we use the natural basis. To unify formulas for the interior and exterior cases, we write $\psi_k(z) = \lambda^{-k}(z)/k$, where $\lambda(z) = z^{-1}$ for the interior problem and $\lambda(z) = z$

for the exterior one. (Correspondingly, z_0 is either 0 or ∞ .) We also employ the notation $f(z_j) = f_j$, $\lambda(z_j) = \lambda_j$,

$$D(z) = \sum_{k \geq 1} \frac{z^{-k}}{k} \frac{\partial}{\partial t_k}, \quad \bar{D}(\bar{z}) = \sum_{k \geq 1} \frac{\bar{z}^{-k}}{k} \frac{\partial}{\partial \bar{t}_k}, \quad (4.4)$$

so that $\mathcal{D}(z) = \partial_{t_0} + D(z) + \bar{D}(\bar{z})$.

Representations of the conformal map through the tau-function can be obtained from (4.3) by separating holomorphic and antiholomorphic parts in z_1 and z_2 of the Green function. Holomorphic parts in both variables yield

$$\log \frac{f_1 - f_2}{\lambda_1 - \lambda_2} = D(\lambda_1)D(\lambda_2)F - \frac{1}{2}\partial_{t_0}^2 F. \quad (4.5)$$

Terms holomorphic in z_1 and antiholomorphic in z_2 yield

$$\log \left(1 - \frac{1}{f_1 f_2} \right) = -D(\lambda_1)\bar{D}(\bar{\lambda}_2)F. \quad (4.6)$$

Various specifications of (4.3), (4.5) and (4.6) lead to representations of the conformal maps through the tau function. Tending, for example, $z_2 \rightarrow z_0$ in (4.3) and extracting holomorphic parts in z_1 , we get the following expressions for the conformal maps through the tau-functions:

$$f(z) = \lambda(z) \exp \left(-\frac{1}{2}\partial_{t_0}^2 F - \partial_{t_0} D(\lambda(z))F \right). \quad (4.7)$$

The leading coefficients as $z \rightarrow z_0$ give formulas for the conformal radius:

$$2 \log r = -\kappa \partial_{t_0}^2 F \quad (4.8)$$

(recall that the sign factor κ is introduced in Sec.2 to distinguish between interior and exterior problems). Limits $z_2 \rightarrow z_0$ in (4.5), (4.6) give other representations for the conformal maps,

$$\begin{aligned} f(z) &= e^{-\frac{1}{2}\partial_{t_0}^2 F} \left(\lambda(z) - \partial_{t_0 t_1}^2 F - D(\lambda(z))\partial_{t_1} F \right), \\ \frac{1}{f(z)} &= e^{-\frac{1}{2}\partial_{t_0}^2 F} D(\lambda(z))\partial_{\bar{t}_1} F \end{aligned} \quad (4.9)$$

and another representation of the conformal radius:

$$r^{-2\kappa} = \partial_{t_1 \bar{t}_1}^2 F. \quad (4.10)$$

These formulas agree with (4.8) provided F satisfies the dispersionless Toda equation

$$\partial_{t_1 \bar{t}_1}^2 F = e^{\partial_{t_0}^2 F}. \quad (4.11)$$

In fact all coefficients of the Taylor expansion of the conformal map around the normalization point can be easily read from (4.9).

Merging the points z_1 and z_2 in (4.5), (4.6), we get formulas for the derivative and the modules of the conformal maps:

$$\begin{aligned} f'(z) &= \lambda'(z) e^{D^2(\lambda(z))F - \frac{1}{2}\partial_{t_0}^2 F} \\ 1 - |f(z)|^{-2} &= \exp\left(-D(\lambda(z))\bar{D}(\overline{\lambda(z)})F\right). \end{aligned} \quad (4.12)$$

It is easy to see that the generators of commuting flows introduced in (3.8) are expressed through the tau-function as follows:

$$\begin{aligned} H_k(z) &= \lambda^k(z) - \left(\frac{1}{2}\partial_{t_0} + D(\lambda(z))\right)\partial_{t_k} F, \\ \tilde{H}_k(z) &= \left(\frac{1}{2}\partial_{t_0} + D(\lambda(z))\right)\partial_{\bar{t}_k} F, \\ H_0(z) &= \log \lambda(z) - \left(\frac{1}{2}\partial_{t_0} + D(\lambda(z))\right)\partial_{t_0} F. \end{aligned}$$

As is seen from (4.3), coefficients of the Laurent series of the regular part of the Green function are second order derivatives of the tau-function. The formula

$$2(z_1\partial_{z_1} + z_2\partial_{z_2})G(z_1, z_2) = \frac{1}{2\pi i} \oint_{\partial D} \frac{d \log f_{z_1}(z) d \log f_{z_2}(z)}{d \log z}$$

can be easily proved by substituting (3.4) and evaluating residues at the poles z_1, z_2 . The left hand side of this formula could be thought of as a dispersionless analog of the Gelfand-Dikii resolvent. Expanding both sides in power series in z_1, z_2 using (4.3) and (3.8), we obtain

$$\begin{aligned} \frac{\partial^2 F}{\partial t_n \partial t_m} &= \frac{1}{2\pi i(n+m)} \oint_{\gamma} \frac{dH_n dH_m}{d \log z}, \quad n \geq 0, \quad m \geq 1 \\ \frac{\partial^2 F}{\partial t_n \partial \bar{t}_m} &= -\frac{1}{2\pi i n} \oint_{\gamma} \frac{dH_n d\bar{H}_m}{d \log z}, \quad n, m \geq 1 \end{aligned}$$

(in the last formula we used (3.10)).

In addition to the relations above, we point out that the tau-function provides new representations for some classical objects of complex analysis. Consider, for example, the *Schwarz derivative* $\mathcal{S}(f, z) = \frac{f'''(z)}{f'(z)} - \frac{3}{2} \left(\frac{f''(z)}{f'(z)} \right)^2$ of the conformal map f . A non-singular part of $\lim_{\zeta \rightarrow z} \partial_z \partial_{\zeta} G(z, \zeta)$ is $\frac{1}{12} \mathcal{S}(f, z)$, so equation (4.3), with $z_0 = \infty$ and $\lambda(z) = z$, yields [3]

$$\mathcal{S}(f, z) = 6(\partial_z D(z))^2 F.$$

where $\partial_z D(z) = -\sum_{k \geq 1} z^{-k-1} \partial_{t_k}$. The reproducing kernel [11] of the (exterior) domain D , $K(z, \bar{\zeta}) = 2\partial_z \partial_{\bar{\zeta}} G(z, \bar{\zeta})$, is given by

$$K(z, \bar{\zeta}) = \partial_z D(z) \partial_{\bar{\zeta}} \bar{D}(\bar{\zeta}) F. \quad (4.13)$$

4.3 Dispersionless Hirota equations

The tau-function obeys an infinite set of non-linear differential equations. They can be obtained by excluding f from the identities of the type (4.5, 4.6) by means of eq. (4.7). Equations obtained in this way are dispersionless limits of the Hirota bilinear relations [18] obeyed by the tau-functions of integrable hierarchies.

For example, consider eq. (4.5) for pairs of points (z_1, z_2) , (z_2, z_3) and (z_3, z_1) . Exponentiating and summing the equations one gets the identity

$$(z_1 - z_2)e^{D(z_1)D(z_2)F} + (z_2 - z_3)e^{D(z_2)D(z_3)F} + (z_3 - z_1)e^{D(z_3)D(z_1)F} = 0. \quad (4.14)$$

obeyed by the tau-function F for both interior and exterior problems. This is the symmetric form of Hirota's equation for the dKP hierarchy. It encodes algebraic relations between the second order derivatives of the tau-function. These relations are obtained on expanding the Hirota equation in powers of z_1, z_2, z_3 and comparing coefficients² The operators $D(z)$ and $\bar{D}(\bar{z})$ are defined in (4.4). Equation (4.14), as well as other Hirota equations given below, are the same for interior and exterior problems.

More general equations obtained in a similar way include derivatives with respect to t_0 and \bar{t}_k . These are equations of the dToda hierarchy:

$$(z_1 - z_2)e^{D(z_1)D(z_2)F} = z_1e^{-\partial_{t_0}D(z_1)F} - z_2e^{-\partial_{t_0}D(z_2)F}, \quad (4.15)$$

$$1 - e^{-D(z_1)\bar{D}(\bar{z}_2)F} = \frac{1}{z_1\bar{z}_2}e^{\partial_{t_0}(\partial_{t_0}+D(z_1)+\bar{D}(\bar{z}_2))F}. \quad (4.16)$$

Note that eq. (4.15) can be obtained from (4.14) in the formal limit $z_3 \rightarrow 0$ if to understand $\lim_{z \rightarrow 0} D(z)$ as $-\partial_{t_0}$. These equations allow one to express the second order derivatives $\partial_{t_m t_n}^2 F$, $\partial_{t_m \bar{t}_n}^2 F$ with $m, n \geq 1$ as certain functions of the derivatives $\partial_{t_0 t_k}^2 F$, $\partial_{t_0 \bar{t}_k}^2 F$. The dispersionless Toda equation (4.11) is the limit of (4.16) as $z_1, z_2 \rightarrow \infty$. Each side of this equation is the squared conformal radius as one can see from (4.8, 4.10).

In the same manner one can derive other equivalent forms of the Hirota equations:

$$z_1(z_3 - z_2)e^{-\mathcal{D}(z_3)(D(z_1)-D(z_3))F} - z_2(z_3 - z_1)e^{-\mathcal{D}(z_3)(D(z_2)-D(z_3))F} = \\ = z_3(z_1 - z_2)e^{(D(z_1)-D(z_3))(D(z_2)-D(z_3))F}, \quad (4.17)$$

$$1 - e^{-(D(z_1)-D(z_3))(\bar{D}(\bar{z}_2)-\bar{D}(\bar{z}_3))F} = \\ = (z_1^{-1} - z_3^{-1})(\bar{z}_2^{-1} - \bar{z}_3^{-1})e^{\mathcal{D}(z_3)(\mathcal{D}(z_3)+D(z_1)-D(z_3)+\bar{D}(\bar{z}_2)-\bar{D}(\bar{z}_3))F}, \quad (4.18)$$

where the operator $\mathcal{D}(z)$ is defined in (4.2). Eqs. (4.17), (4.18) may be interpreted as analogs of eqs. (4.15, 4.16) with the normalization point moved to z_3 , in which case ∂_{t_0} should be substituted by $\mathcal{D}(z_3)$. In the limit $z_3 \rightarrow \infty$ they convert into eqs. (4.15), (4.16) respectively. In (4.18) \bar{z}_3 can be regarded as an independent formal variable, not necessarily complex conjugate to z_3 . We stress that the full set of differential equations for F obtained by expanding the Hirota equations (4.15, 4.16) or (4.17, 4.18) is the same.

The special case of (4.18) $z_2 = z_1$ is worth mentioning:

$$1 - e^{-(D(z)-D(\zeta))(\bar{D}(\bar{z})-\bar{D}(\bar{\zeta}))F} = |z^{-1} - \zeta^{-1}|^2 e^{\mathcal{D}(\zeta)\mathcal{D}(z)F}.$$

²In [19, 20], the Hirota equation for the dKP hierarchy was obtained in an equivalent but less symmetric form. It follows from (4.14) in the limit $z_3 \rightarrow \infty$.

Further specialization $\zeta \rightarrow z$ yields

$$|z|^4 \partial_z D(z) \partial_{\bar{z}} \bar{D}(\bar{z}) F = e^{\mathcal{D}^2(z)F}, \quad (4.19)$$

This equation is especially remarkable. On the one hand, it looks like the dispersionless Toda equation (4.11) where ∂_{t_0} , ∂_{t_1} , $\partial_{\bar{t}_1}$ are replaced by $\mathcal{D}(z)$, $z^2 \partial_z D(z)$, $\bar{z}^2 \partial_{\bar{z}} \bar{D}(\bar{z})$ respectively (moreover, (4.19) becomes (4.11) as $z \rightarrow \infty$). On the other hand, (4.19) is equivalent to the Liouville equation. Indeed, eq. (4.19) tells us that the field

$$\chi(z) = \mathcal{D}^2(z)F - \log |z|^4$$

obeys the Liouville equation

$$\partial_z \partial_{\bar{z}} \chi = 2e^\chi$$

for $z \in D$. (Here we imply that D contains ∞ , but similar equations can be written for the interior problem, too.) By virtue of (4.3), (4.12), the solution to this equation can be written as follows:

$$\chi(z) = 2 \lim_{z' \rightarrow z} [G(z, z') - \log |z - z'|]$$

or

$$e^{\chi(z)} = \frac{|f'(z)|^2}{(|f(z)|^2 - 1)^2}$$

Note that $ds^2 = e^\chi dz d\bar{z}$ is the pull back of the Poincare metric with constant negative curvature $R_{ds^2} = -2e^{-\chi} \partial_z \partial_{\bar{z}} \chi = -4$ in D . We also note that the Liouville field χ equals the value of the reproducing kernel [11] of the domain D at merging points: $\chi(z) = \log K(z, \bar{z})$, where $K(z, \bar{z})$ is the reproducing kernel (4.13).

4.4 Residue formulas

In this section we present formulas for third order derivatives of F . Formulas of this type are known in the theory of dispersionless integrable hierarchies and are referred to as residue formulas [12]. They are used, in particular, to prove the associativity equations for tau-functions of Whitham hierarchies [21, 22, 23].

The basic relation to derive the residue formulas is (3.2). By virtue of (4.1), its left hand side is a generating function for third order derivatives of the tau-function. Let us first rewrite this formula in holomorphic terms. To do that we note that $\partial_n G(z, \xi) = -2|\partial_\xi G(z, \xi)|$ for ξ on the boundary (the vector $\text{grad } G$ is normal to the boundary). Further, since G vanishes on the boundary, $\partial_\xi G(z, \xi) d\xi + \partial_{\bar{\xi}} G(z, \xi) d\bar{\xi} = 0$. Therefore, $|\partial_\xi G d\xi| = -\kappa i \partial_\xi G d\xi$ holds on the boundary. Taking all this into account, we rewrite (3.2) as

$$\nabla(a) \nabla(b) \nabla(c) F = \frac{4}{\pi i} \oint_\gamma \frac{\partial_z G(a, z) \partial_z G(b, z) \partial_z G(c, z)}{dz d\bar{z}} (dz)^3.$$

Finally, using $G(a, z) = -\log |f_a(z)|$, where $f_a(z)$ is a holomorphic functions of z (see (3.6)), we get

$$\nabla(a) \nabla(b) \nabla(c) F = -\frac{1}{2\pi i} \oint_\gamma \frac{d \log f_a(z) d \log f_b(z) d \log f_c(z)}{dz d\bar{z}}.$$

Let us expand the both sides in the basis ψ_k in each argument using (2.2), (3.8). Comparing the coefficients, we come to the residue formulas for third order derivatives of the tau-function:

$$\begin{aligned}\frac{\partial^3 F}{\partial t_l \partial t_m \partial t_n} &= -\frac{1}{2\pi i} \oint_{\gamma} \frac{dH_l dH_m dH_n}{dz d\bar{z}}, \quad l, m, n \geq 0, \\ \frac{\partial^3 F}{\partial t_l \partial t_m \partial \bar{t}_n} &= \frac{1}{2\pi i} \oint_{\gamma} \frac{dH_l dH_m d\bar{H}_n}{dz d\bar{z}}, \quad l, m \geq 0, \quad n \geq 1.\end{aligned}\tag{4.20}$$

The formulas (4.20) were first obtained by I.Krichever [9] by other means.

5 Integral representations of the tau-function

In this section we discuss integral representations of the tau-functions and connection between the tau-functions for the interior and exterior Dirichlet problems. We employ the special notation D_{int} for the interior domain bounded by the curve γ and D_{ext} for the exterior one. We use the natural basis, i.e., the normalization points are $0 \in D_{\text{int}}$ and $\infty \in D_{\text{ext}}$, and the basis functions are z^k/k and z^{-k}/k respectively. At last, the tau-functions are denoted as F^{int} , F^{ext} .

5.1 Electrostatic potentials

Let us begin with the exterior problem. Our starting point is eq. (4.3) which we rewrite here introducing the auxiliary function (potential)

$$\Phi^{\text{ext}}(z) = \mathcal{D}(z) F^{\text{ext}}.\tag{5.1}$$

Eq. (4.3) then reads

$$\mathcal{D}(z) \Phi^{\text{ext}}(\xi) = 2G(z, \xi) - 2 \log |z^{-1} - \xi^{-1}|$$

for both z and ξ being in D_{ext} . (We recall that $\mathcal{D}(z) = \partial_{t_0} + D(z) + \bar{D}(\bar{z})$.) This function admits an electrostatic interpretation explained below. When one of the points reaches the boundary, the Green function vanishes and one has

$$\mathcal{D}(\xi) \Phi^{\text{ext}}(z) = -2 \log |z^{-1} - \xi^{-1}|, \quad \xi \in \gamma.$$

The last formula makes sense of the variation of the function Φ^{ext} under the bump deformation of the domain (see sect. 2). According to (2.3), the variation is

$$\delta_{\epsilon(\xi)} \Phi^{\text{ext}}(z) = \frac{2\epsilon}{\pi} \log |z^{-1} - \xi^{-1}|$$

and, therefore, one may write the integral representation

$$\Phi^{\text{ext}}(z) = -\frac{2}{\pi} \int_{D_{\text{int}}} d^2 \zeta \log |z^{-1} - \zeta^{-1}|\tag{5.2}$$

up to a harmonic function that do not depend on shape of the domain. Adding to (5.2) harmonic functions independent of times amounts to redefinition of the tau-function by terms linear in times, which contribute neither to the Green functions nor to the formulas for conformal maps (4.7), (4.9), and leave Hirota equations unchanged. We set all such terms to vanish. In a similar way, for the interior problem we get $\delta_{\epsilon(\xi)}\Phi^{\text{int}}(z) = \frac{2\epsilon}{\pi} \log |z - \xi|$ and

$$\Phi^{\text{int}}(z) = \frac{2}{\pi} \int_{D_{\text{ext}}} d^2\zeta \log |z - \zeta| - c,$$

where the constant c is necessary for regularization of the divergent integral over D_{ext} . Let us cut off the integral at a big circle of radius R and set $c = c(R) = \frac{2}{\pi} \int_{|\zeta| < R} d^2\zeta \log |\zeta| = R^2 \log R^2 - R^2$. Then we get

$$\Phi^{\text{int}}(z) = \lim_{R \rightarrow \infty} \left(\frac{2}{\pi} \int_{D_{\text{ext}}} d^2\zeta \log |z - \zeta| - c(R) \right) = \frac{2}{\pi} \int_{D_{\text{ext}}} d^2\zeta \log |1 - z\zeta^{-1}| - v_0, \quad (5.3)$$

where we introduced

$$v_0 = \frac{1}{\pi} \int_{D_{\text{int}}} \log |z|^2 d^2z.$$

Formulas (5.2), (5.3) present the potentials Φ^{ext} , Φ^{int} in the form of an integral over the *complementary* domain, since by definition (5.1) the functions Φ^{ext} , Φ^{int} are harmonic in D_{ext} , D_{int} respectively. One may use these formulas to extend the functions Φ to the whole complex plane. For example, (5.2) defines a function Φ^{ext} which is harmonic in the exterior domain and satisfies the Poisson equation $-\partial_z \partial_{\bar{z}} \Phi^{\text{ext}}(z) = 1 - \pi t_0 \delta(z)$ in D_{int} . This function is the potential generated by a uniformly distributed charge in D_{int} and a compensating point-like charge at the origin. The expansions of Φ^{ext} at small and large z read:

$$\begin{aligned} \Phi^{\text{ext}}(z) &= -|z|^2 + 2t_0 \log |z| + \sum_{k>0} (t_k^{\text{ext}} z^k + \bar{t}_k^{\text{ext}} \bar{z}^k), \quad z \rightarrow 0, \\ \Phi^{\text{ext}}(z) &= v_0 + \sum_{k>0} (t_k^{\text{int}} z^{-k} + \bar{t}_k^{\text{int}} \bar{z}^{-k}), \quad z \rightarrow \infty, \end{aligned} \quad (5.4)$$

where the additional superscripts are set to distinguish between exterior and interior moments. Under our assumptions, these series converge in D_{int} and D_{ext} respectively. The functions Φ^{ext} and $\partial_z \Phi^{\text{ext}}$ are continuous at the boundary.

5.2 Integral formulas for the tau-function

Using the same strategy, we set z in (5.1) to the boundary and interpret this formula as a result of the bump deformation of the domain. It is easy to check that the variation of

$$F^{\text{ext}} = \frac{1}{2\pi} \int_{D_{\text{int}}} d^2z \Phi^{\text{ext}}(z) = -\frac{1}{\pi^2} \int_{D_{\text{int}}} d^2z \int_{D_{\text{int}}} d^2\zeta \log \left| \frac{1}{z} - \frac{1}{\zeta} \right| \quad (5.5)$$

is

$$\begin{aligned} \delta_{\epsilon(\xi)} F^{\text{ext}} &= \frac{\epsilon}{2\pi} \Phi^{\text{ext}}(\xi) + \frac{1}{2\pi} \int_{D_{\text{int}}} d^2z \delta_{\epsilon(\xi)} \Phi^{\text{ext}}(z) = \\ &= \frac{\epsilon}{2\pi} \Phi^{\text{ext}}(\xi) - \frac{\epsilon}{\pi^2} \int_{D_{\text{int}}} d^2z \log |z^{-1} - \xi^{-1}| = \frac{\epsilon}{\pi} \Phi^{\text{ext}}(\xi). \end{aligned}$$

Eq. (5.5) presents the tau-function for the exterior Dirichlet problem as a double integral over the domain complement to the D_{ext} [8].

Similar arguments give the tau-function of the interior problem:

$$F^{\text{int}} = \lim_{R \rightarrow \infty} \left(-\frac{1}{\pi^2} \int_{D_{\text{ext}}} d^2 z \int_{D_{\text{ext}}} d^2 \zeta \log |z - \zeta| + C(R) - c(R)t_0 \right)$$

with $c(R)$ as in (5.3), and

$$C(R) = \frac{1}{\pi^2} \int_{|z| < R} d^2 z \int_{|\zeta| < R} d^2 \zeta \log |z - \zeta| = \frac{1}{4} R^4 (2 \log R^2 - 1).$$

It is implied that the integral over D_{ext} is cut off at R . Note the relation

$$F^{\text{int}} + F^{\text{ext}} = \frac{1}{\pi^2} \int_{D_{\text{int}}} d^2 z \int_{D_{\text{ext}}} d^2 \zeta \log |1 - z\zeta^{-1}|^2 \quad (5.6)$$

In fact, the tau-functions admit other useful integral representations. Let us mention some of them. In terms of the potential Φ^{ext} , extended to the whole complex plane as explained above, the tau-function of the exterior problem may be represented as a (regularized) integral over the whole complex plane:

$$F^{\text{ext}} = \lim_{\varepsilon \rightarrow 0} \left(\frac{1}{2\pi} \int_{\mathbb{C} \setminus D_\varepsilon} \left| \partial_z \Phi^{\text{ext}} \right|^2 d^2 z + t_0^2 \log \varepsilon \right) \quad (5.7)$$

where D_ε is a small disk of radius ε centered at the origin. In the electrostatic interpretation, the tau-function is basically the energy of the system of charges mentioned above. Stokes formula gives an integral representation through a contour integral:

$$F^{\text{ext}} = -\frac{1}{4} t_0^2 + \frac{1}{2\pi} \oint_\gamma \Phi^{\text{ext}}(z) \frac{\bar{z} dz - z d\bar{z}}{4i} \quad (5.8)$$

Similar formulas can be written for the interior problem.

5.3 Legendre transform $F^{\text{int}} \leftrightarrow F^{\text{ext}}$

The tau-functions for the interior and exterior problems are connected by a Legendre transform. The former is the function of the interior moments t_k^{int} and the area t_0 , while the latter is the function of the exterior moments t_k^{ext} and the area. From (5.1) we read that first order derivatives of the tau-function for the Dirichlet problem in D_{int} or D_{ext} with respect to the harmonic moments are (up to the factor k) harmonic moments of the complimentary domain:

$$\frac{\partial F^{\text{int}}}{\partial t_k^{\text{int}}} = k t_k^{\text{ext}}, \quad \frac{\partial F^{\text{ext}}}{\partial t_k^{\text{ext}}} = k t_k^{\text{int}} \quad (5.9)$$

and

$$\frac{\partial F^{\text{ext}}}{\partial t_0} = -\frac{\partial F^{\text{int}}}{\partial t_0} = v_0.$$

Using (5.6), we obtain the relation

$$F^{\text{int}}(t_0, \mathbf{t}^{\text{int}}, \bar{\mathbf{t}}^{\text{int}}) = \sum_{k=1}^{\infty} \left(k t_k^{\text{int}} t_k^{\text{ext}} + k \bar{t}_k^{\text{int}} \bar{t}_k^{\text{ext}} \right) - F^{\text{ext}}(t_0, \mathbf{t}^{\text{ext}}, \bar{\mathbf{t}}^{\text{ext}}).$$

By virtue of (5.9), it means that the functions F^{int} and F^{ext} are related by Legendre transforms with respect to all variables but t_0 :

$$\begin{aligned} F^{\text{int}} &= \sum_{k=1}^{\infty} \left(t_k^{\text{ext}} \frac{\partial F^{\text{ext}}}{\partial t_k^{\text{ext}}} + \bar{t}_k^{\text{ext}} \frac{\partial F^{\text{ext}}}{\partial \bar{t}_k^{\text{ext}}} \right) - F^{\text{ext}}, \\ F^{\text{ext}} &= \sum_{k=1}^{\infty} \left(t_k^{\text{int}} \frac{\partial F^{\text{int}}}{\partial t_k^{\text{int}}} + \bar{t}_k^{\text{int}} \frac{\partial F^{\text{int}}}{\partial \bar{t}_k^{\text{int}}} \right) - F^{\text{int}}. \end{aligned} \tag{5.10}$$

5.4 Homogeneity properties of the tau-function

Expansion of the potential $\Phi^{\text{ext}}(z)$ around the origin (5.4) allows one to prove another important property of the tau-functions. Integrating both sides of it over D_{int} and using (5.9), we obtain a quasihomogeneity condition for F^{ext} . A similar condition for F^{int} is most easily derived from (5.10). They are:

$$\begin{aligned} 4F^{\text{int}} &= t_0^2 + 2t_0 \partial_{t_0} F^{\text{int}} + \sum_{k>0} (2+k)(t_k \partial_{t_k} F^{\text{int}} + \bar{t}_k \partial_{\bar{t}_k} F^{\text{int}}), \\ 4F^{\text{ext}} &= -t_0^2 + 2t_0 \partial_{t_0} F^{\text{ext}} + \sum_{k>0} (2-k)(t_k \partial_{t_k} F^{\text{ext}} + \bar{t}_k \partial_{\bar{t}_k} F^{\text{ext}}). \end{aligned} \tag{5.11}$$

These formulas reflect the scaling of moments as $z \rightarrow \lambda z$ with real λ : $t_k^{\text{ext}} \rightarrow \lambda^{2+k} t_k^{\text{ext}}$, $t_k^{\text{int}} \rightarrow \lambda^{2-k} t_k^{\text{int}}$. The logarithmic moment v_0 , under the same rescaling, exhibits a more complicated behaviour: $v_0 \rightarrow \lambda^2 v_0 + t_0 \lambda^2 \log \lambda^2$. To get rid of the “anomaly term” t_0^2 one may modify the tau-function by subtracting $\frac{1}{4} t_0^2 \log t_0^2$.

6 Dirichlet problem on the plane with a gap

Consider the case when the domain D_{int} shrinks to a segment of the real axis. Then the interior Dirichlet problem does not seem to make sense anymore but the exterior one is still well-posed: find a bounded harmonic function in the complex plane such that it equals a given function on the segment. The problem admits an explicit solution (see e.g. [24]). Possible variations of the data are variation of the function on the segment and the endpoints of the segment. Solution of this problem as well as its integrable structure may be obtained from the formulas for a smooth domain as a result of a singular limit when a smooth domain shrinks to the segment.

The tau-function, obtained in this way, is a partition function of the Hermitean one-matrix model for the one-cut solution in the planar large N limit [25].

6.1 Shrinking the domain: the limiting procedure

Let us consider a family of curves $\gamma(\varepsilon)$ obtained from a given curve γ by rescaling of the y -axis as $y \rightarrow \varepsilon y$. (If the curve γ is given by an equation $P(x, y) = 0$, then $\gamma(\varepsilon)$ is given by $P(x, y/\varepsilon) = 0$.) We are interested in the limit $\varepsilon \rightarrow 0$, $\gamma(0)$ being a segment of the

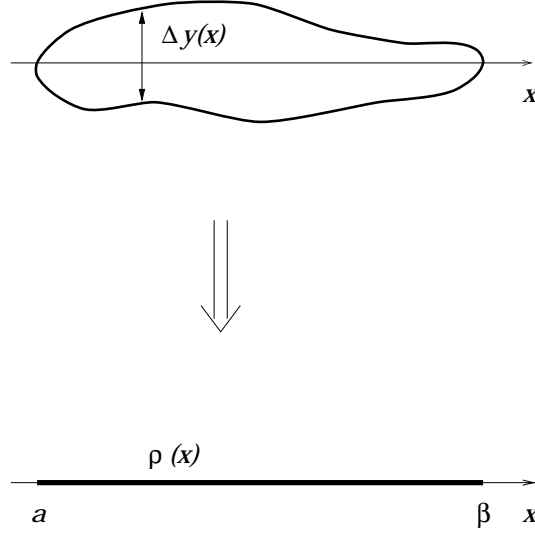


Figure 2: A thin domain stretched along the real axis with thickness $\Delta y(x)$ shrinks into a segment with density $\rho(x)$.

real axis. We denote the endpoints by α, β . Let $\Delta y(x)$ be the thickness of the domain bounded by the curve $\gamma(\varepsilon)$ at the point x (see Fig. 2).

We introduce the function

$$\rho(x) = \lim_{\varepsilon \rightarrow 0} \frac{\Delta y(x)}{\varepsilon}.$$

It is easy to see that in case of general position this function can be represented as

$$\rho(x) = \sqrt{(x - \alpha)(\beta - x)} M(x), \quad (6.1)$$

where $M(x)$ is a smooth function regular at the edges. So, instead of the space of contours γ we have the space of real positive functions $\rho(x)$ of the form (6.1) with a finite support $[\alpha, \beta]$ (the endpoints of which are not fixed but may vary).

Let us use the first equation in (5.4) to define times as coefficients of the expansion of the potential $\Phi(x) = \Phi^{\text{ext}}(x)$ generated by the thin domain with a uniform charge density (and with a point-like charge at the origin). In the leading order in ε the rescaled potential is

$$\phi(x) \equiv \lim_{\varepsilon \rightarrow 0} \frac{\Phi(x)}{\varepsilon} = -\frac{1}{\pi} \int_{\alpha}^{\beta} dx' \rho(x') \log\left(\frac{1}{x} - \frac{1}{x'}\right)^2 = T_0 \log x^2 + \sum_{k \geq 1} T_k x^k. \quad (6.2)$$

Comparing this with (5.4), we get

$$\begin{aligned} T_0 &= \lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} t_0, \\ T_k &= \lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} (t_k^{\text{ext}} + \bar{t}_k^{\text{ext}}), \quad k \geq 1, k \neq 2, \\ T_2 &= \lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} (t_2^{\text{ext}} + \bar{t}_2^{\text{ext}} - 1). \end{aligned} \quad (6.3)$$

Note that $T_0 = \frac{1}{\pi} \int_{\alpha}^{\beta} \rho(x) dx$ but similar integral representations for other times, $T_k = \frac{2}{\pi k} \int_{\alpha}^{\beta} \rho(x) x^{-k} dx$, which formally follow from (6.2), are ill-defined. On the other hand,

harmonic moments of the interior behave, in the scaling limit, as $t_k^{\text{int}} = \varepsilon k^{-1} \mu_k + O(\varepsilon^2)$, where μ_k are well-defined moments of the function ρ on the segment:

$$\mu_k = \frac{1}{\pi} \int_{\alpha}^{\beta} \rho(x) x^k dx. \quad (6.4)$$

Using integral formula (5.5), it is now straightforward to find the scaling limit of the tau-function F^{ext} for the exterior problem. Taking into account (6.3), we introduce the function F^{cut} as follows:

$$\begin{aligned} F^{\text{ext}}(\varepsilon t_0; \varepsilon t_1, \tfrac{1}{2} + \varepsilon t_2, \varepsilon t_3, \dots; \varepsilon \bar{t}_1, \tfrac{1}{2} + \varepsilon \bar{t}_2, \varepsilon \bar{t}_3, \dots) = \\ = \varepsilon^2 F^{\text{cut}}(T_0; T_1, T_2, T_3, \dots) + O(\varepsilon^3), \end{aligned} \quad (6.5)$$

where $T_0 = t_0$, $T_k = t_k + \bar{t}_k$. Since the second order derivatives of F^{ext} are invariant under the rescaling (6.5), the function F^{cut} obeys the Hirota equations (4.14)–(4.16) (as F^{ext} does) where one has to set $\partial_{t_k} = \partial_{\bar{t}_k} = \partial_{T_k}$. The latter means that F^{cut} is a solution of the reduced dToda hierarchy (see e.g. [12]). This sort of reduction is usually referred to as dispersionless Toda chain. We conjecture that other types of reduction correspond, in the same way, to shrinking of D_{int} to slit domains of a more complicated form.

An integral representation for the F^{cut} (obtained as a limit of (5.5)) reads:

$$F^{\text{cut}} = -\frac{1}{\pi^2} \int_{\alpha}^{\beta} \int_{\alpha}^{\beta} \rho(x_1) \rho(x_2) \log |x_1^{-1} - x_2^{-1}| dx_1 dx_2$$

Represent this as $F^{\text{cut}} = \frac{1}{2\pi} \int_{\alpha}^{\beta} \rho(x) \phi(x) dx = \frac{1}{2} \sum_{k \geq 0} \mu_k T_k$, where μ_k are defined in (6.4)

and

$$\mu_0 = \frac{1}{\pi} \int_{\alpha}^{\beta} \rho(x) \log x^2 dx.$$

It is clear from the limit of (5.9) that $\mu_k = \partial_{T_k} F^{\text{cut}}$ for $k \geq 0$. Therefore, we obtain the relation

$$2F^{\text{cut}} = \sum_{k \geq 0} T_k \partial_{T_k} F^{\text{cut}}, \quad (6.6)$$

which means that F^{cut} is a homogeneous function of degree 2. This formula also means that the tau-function F^{cut} is “self-dual” under the Legendre transform with respect to T_0, T_1, T_2, \dots . However, the analog of the Legendre transform (5.10) we discussed in sect. 5.3 does not include T_0 , so the analog of the function F^{int} is the function

$$\begin{aligned} E^{\text{cut}} &= \sum_{k \geq 1} T_k \mu_k - F^{\text{cut}} = F^{\text{cut}} - T_0 \frac{\partial F^{\text{cut}}}{\partial T_0} = \\ &= -\frac{1}{\pi^2} \int_{\alpha}^{\beta} \int_{\alpha}^{\beta} \rho(x_1) \rho(x_2) \log |x_1 - x_2| dx_1 dx_2, \end{aligned}$$

which is the electrostatic energy of the segment with the charge density ρ on it, regarded as a function of the variables T_0, μ_1, μ_2, \dots . Properties of this function and its possible relation to the Hamburger 1D moment problem are to be further investigated.

6.2 Conformal maps

The conformal map $f : \mathbf{C} \setminus [\alpha, \beta] \longrightarrow E$ from the plane with the gap onto the exterior E of the unit disk is given by the explicit formula

$$f(z) = \frac{2z - \alpha - \beta + 2\sqrt{(z - \alpha)(z - \beta)}}{\beta - \alpha}.$$

All formulas which connect conformal maps and tau-function remain true in this case, too. In particular, expanding $f(z)$ as $z \rightarrow \infty$, $f(z) = \frac{4z}{\beta - \alpha} - \frac{2(\alpha + \beta)}{\beta - \alpha} + O(z^{-1})$, we read from (4.9) formulas for the endpoints of the cut:

$$\beta - \alpha = 4 \exp \left(\frac{1}{2} \partial_{T_0}^2 F^{\text{cut}} \right), \quad \beta + \alpha = 2 \partial_{T_0 T_1}^2 F^{\text{cut}}.$$

The Green function is expressed through the $f(z)$ by the same formula (1.2).

Set $w = f(z)$, then the inverse map is

$$z(w) = \frac{\beta - \alpha}{4}(w + w^{-1}) + \frac{\beta + \alpha}{2}.$$

Clearly, $z(w) = \bar{z}(w^{-1})$, so the constraint on the Lax functions is now $L(w) = \tilde{L}(w)$ which signifies the reduction to the dispersionless Toda chain.

6.3 Relation to matrix models

In analogy with variations of the data of the Dirichlet problem discussed in Sec. 3, we can consider the following problem: given a set of T_k , $k \geq 0$, to find endpoints α, β of the segment and the density $\rho(x)$ as functions of these parameters. This problem has been appeared in studies of the planar $N \rightarrow \infty$ limit of the Hermitean matrix model. In this case the function $\rho(x)$ is a density of eigenvalues [25]. For completeness, we recall the basic points.

Taking derivative of (6.2), we get the equation which connects the set of T_k with α, β and $\rho(x)$:

$$\frac{2}{\pi} \text{v.p.} \int_{\alpha}^{\beta} \frac{\rho(x') dx'}{x' - x} = V'(x),$$

where

$$V(x) = \phi(x) - T_0 \log x^2 = \sum_{k \geq 1} T_k x^k. \quad (6.7)$$

Consider the function

$$W(z) = \frac{1}{\pi} \int_{\alpha}^{\beta} \frac{\rho(x) dx}{z - x}.$$

It is analytic in the complex plane with the cut $[\alpha, \beta]$. As $z \rightarrow \infty$, it behaves as $W(z) = T_0 z^{-1} + O(z^{-2})$. On the cut, its boundary values are $W(x \pm i0) = -\frac{1}{2} V'(x) \mp i\rho(x)$. The function $W(z)$ is uniquely defined by these analytic properties. So, to find ρ one should

find the holomorphic function from its mean value on the cut. The result is given by the explicit formula

$$W(z) = \frac{1}{4\pi i} \oint \frac{d\zeta V'(\zeta)}{\zeta - z} \frac{\sqrt{(z - \alpha)(z - \beta)}}{\sqrt{(\zeta - \alpha)(\zeta - \beta)}}, \quad (6.8)$$

where the contour encircles the cut but not the point z . The endpoints of the cut are fixed by comparing the leading terms of (6.8) with the required asymptote of the $W(z)$ as $z \rightarrow \infty$. This leads to the hodograph-like formulas

$$\frac{1}{2\pi i} \oint \frac{V'(z)dz}{\sqrt{(z - \alpha)(z - \beta)}} = 0, \quad \frac{1}{2\pi i} \oint \frac{zV'(z)dz}{\sqrt{(z - \alpha)(z - \beta)}} = -2T_0,$$

which implicitly determine α, β as functions of T_k .

It follows from the above that F^{cut} coincides with the one-cut free energy of the Hermitean one-matrix model with potential (6.7) in the planar large N limit. (As a matter of fact, it was shown in [26] that the partition function of the matrix model at finite N is the tau-function of the Toda chain hierarchy with dispersion.) Combining (6.6) with the $\varepsilon \rightarrow 0$ limit of the second relation in (5.11), we obtain

$$\sum_{k \geq 1} k T_k \partial_{T_k} F^{\text{cut}} + T_0^2 = 0.$$

This identity is known as the Virasoro L_0 -constraint on the dispersionless tau-function F^{cut} [26, 12]. Other Virasoro constraints can be obtained in the $\varepsilon \rightarrow 0$ limit from the $W_{1+\infty}$ -constraints on the tau-function F^{ext} . We do not discuss them here.

6.4 An example: Gaussian matrix model

If only first three variables are nonzero (i.e., $T_0, T_1, T_2 \neq 0$) function F^{cut} can be found explicitly. Consider a family of ellipses with axes l and εs centered at some point x_0 on real axis:

$$\frac{(x - x_0)^2}{l^2} + \frac{y^2}{\varepsilon^2 s^2} = \frac{1}{4}.$$

The harmonic moments, as $\varepsilon \rightarrow 0$, are (see Appendix in ref. [3]): $t_0 = \frac{1}{4}\varepsilon s l$, $t_1^{\text{ext}} = 2\varepsilon x_0 s l^{-1} + O(\varepsilon^2)$, $t_2^{\text{ext}} = \frac{1}{2} - \varepsilon s l^{-1} + O(\varepsilon^2)$ and all other moments of the complement to ellipse vanish. From (6.3) we read values of the rescaled variables:

$$T_0 = \frac{1}{4}(\beta - \alpha)s, \quad T_1 = 2 \frac{\beta + \alpha}{\beta - \alpha} s, \quad T_2 = -\frac{2}{\beta - \alpha} s,$$

and all the rest are zero. Here we have expressed times through the endpoints $\alpha = x_0 - \frac{1}{2}l$, $\beta = x_0 + \frac{1}{2}l$ of the segment and the extra parameter s . The density function is

$$\rho(x) = -T_2 \sqrt{(\beta - x)(x - \alpha)}.$$

(Note that $T_2 < 0$.)

Using explicit form of the tau-function for ellipse [3, 8],

$$F^{\text{ext}} \Big|_{\text{ellipse}} = \frac{1}{2} t_0^2 \log \frac{t_0}{1 - 4t_2 \bar{t}_2} - \frac{3}{4} t_0^2 + t_0 \frac{t_1 \bar{t}_1 + t_1^2 \bar{t}_2 + \bar{t}_1^2 t_2}{1 - 4t_2 \bar{t}_2},$$

and the scaling procedure (6.5), we find

$$F^{\text{cut}}(T_0, T_1, T_2, 0, 0, \dots) = \frac{1}{2}T_0^2 \log\left(\frac{T_0}{-2T_2}\right) - \frac{3}{4}T_0^2 - \frac{T_0T_1^2}{4T_2}.$$

This is expression for the free energy of the Gaussian matrix model in the planar large N limit [25]. The same result can be obtained from the integral formula for F^{cut} .

Acknowledgments

We acknowledge useful discussions with A.Boyarsky, L.Chekhov, B.Dubrovin, P.Di Francesco M.Mineev-Weinstein, V.Kazakov, A.A.Kirillov, I.Kostov, A.Polyakov, O.Ruchay-skiy, and especially with I.Krichever and L.Takhtajan. The work of A.M. and A.Z. was partially supported by CRDF grant RP1-2102 and RFBR grant No. 00-02-16477, P.W. and A.Z. have been supported by grants NSF DMR 9971332 and MRSEC NSF DMR 9808595. A.M. was partially supported by INTAS grant 97-0103 and grant for support of scientific schools No. 00-15-96566. A.Z. was partially supported by grant INTAS-99-0590, by RFBR grant 98-01-00344 and grant for support of scientific schools No. 00-15-96557.

References

- [1] A. Hurwitz and R. Courant, *Vorlesungen über allgemeine Funktionentheorie und elliptische Funktionen. Herausgegeben und ergänzt durch einen Abschnitt über geometrische Funktionentheorie*, Springer-Verlag, 1964 (Russian translation, adapted by M.A. Evgrafov: *Theory of functions*, Nauka, Moscow, 1968).
- [2] M. Mineev-Weinstein, P.B. Wiegmann and A. Zabrodin, Phys. Rev. Lett. **84** (2000) 5106, e-print archive: nlin.SI/0001007.
- [3] P.B. Wiegmann and A. Zabrodin, Commun. Math. Phys. **213** (2000) 523, e-print archive: hep-th/9909147.
- [4] A. Hanany, Y. Oz and R. Plesser, Nucl. Phys. **B425** (1994) 150-172;
K. Takasaki, Commun. Math. Phys. **170** (1995) 101-116;
T. Eguchi and H. Kanno, Phys. Lett. **331B** (1994) 330.
- [5] J.M.Daul, V.A.Kazakov and I.K.Kostov, Nucl. Phys. **B409** (1993) 311-338;
L.Bonora and C.S.Xiong, Phys. Lett. **B347** (1995) 41-48.
- [6] J. Gibbons and S. P. Tsarev, Phys. Lett. **211A** (1996) 19-24; *ibid* **258A** (1999) 263.
- [7] J. Hadamard, *Mém. présentés par divers savants à l'Acad. sci.*, **33** (1908).
- [8] I.K. Kostov, I.M. Krichever, M. Mineev-Weinstein, P.B. Wiegmann and A. Zabrodin, *τ -function for analytic curves*, Random matrices and their applications, MSRI publications, vol.40, Cambridge Academic Press, 2001, e-print archive: hep-th/0005259.
- [9] I. Krichever, unpublished.

- [10] L. Takhtajan, *Free bosons and tau-functions for compact Riemann surfaces and closed smooth Jordan curves. I. Current correlation functions*, e-print archive: math.QA/0102164.
- [11] E. Hille, *Analytic function theory*, v.II, Ginn and Company, 1962.
- [12] I.M. Krichever, *Funct. Anal Appl.* **22** (1989) 200-213;
Commun. Pure. Appl. Math. **47** (1992) 437, e-print archive: hep-th/9205110.
- [13] T. Takebe, *Adv. Series in Math. Phys.* **16** (1992), Proceedings of RIMS Research Project 1991, 923-940.
- [14] T. Shiota, *Invent. Math.* **83** (1986) 333-382.
- [15] M. Douglas, *Phys. Lett.* **B238** (1990) 176.
- [16] M. Mineev-Weinstein and A. Zabrodin, Proceedings of the Workshop NEEDS 99 (Crete, Greece, June 1999), e-print archive: solv-int/9912012.
- [17] S. Kharchev, A. Marshakov, A. Mironov and A. Morozov, *Mod. Phys. Lett.* **A8** (1993) 1047-1061, e-print archive: hep-th/9208046.
- [18] M. Sato, *Soliton Equations and Universal Grassmann Manifold* Math. Lect. Notes Ser., Vol. 18, Sophia University, Tokyo (1984);
E. Date, M. Jimbo, M.Kashiwara and T. Miwa, *Transformation groups for soliton equations*, in: Nonlinear Integrable Systems, eds. M. Jimbo and T. Miwa, Singapore, World Scientific, 1983.
- [19] J. Gibbons and Y. Kodama, Proceedings of NATO ASI “Singular Limits of Dispersive Waves”, ed. N. Ercolani, London – New York, Plenum, 1994;
R. Carroll and Y. Kodama, *J. Phys. A: Math. Gen.* **A28** (1995) 6373.
- [20] K. Takasaki and T. Takebe, *Rev. Math. Phys.* **7** (1995) 743-808.
- [21] R. Dijkgraaf, E. Verlinde and H. Verlinde, *Nucl. Phys.* **B352** (1991) 59.
- [22] A. Marshakov, A. Mironov and A. Morozov, *Phys. Lett.* **B389** (1996) 43-52, e-print archive: hep-th/9607109.
- [23] A. Boyarsky, A. Marshakov, O. Ruchayskiy, P. Wiegmann and A. Zabrodin, to be published in *Phys. Lett. B*, e-print archive: hep-th/0105260.
- [24] F.Gakhov, *Boundary problems*, Nauka, Moscow, 1977 (in Russian);
A.Bitsadze, *Foundations of the theory of analytic functions of a complex variable*, Nauka, Moscow, 1984 (in Russian)
- [25] E. Brézin, C. Itzykson, G. Parisi and J.-B. Zuber, *Commun. Math. Phys.* **59** (1978) 35.
- [26] A. Gerasimov, A. Marshakov, A. Mironov, A. Morozov and A. Orlov, *Nucl. Phys.* **B357** (1991) 565-618.